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A REMARK ON FLOW INVARIANCE FOR SEMILINEAR PARABOLIC EQUATIONS

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ABSTRACT

It is shown that a flow invariance problem for semilinear parabolic equations is locally soluble if and only if Pavel's tangent condition holds.

1. Introduction

Motivated by [1],[7],[8],[10], we consider the local existence of solutions to the parabolic evolution equations

$$(1.1)\qquad \begin{cases} du/dt + A(t)u = f(t,u), & s < t \leqq T, \\ u(s) = u_0, \end{cases}$$

associated with the flow invariance condition

$$(1.2)\qquad u(t) \in \mathbf{D} \cap D((A(t)+z_0)^a), \qquad s \leqq t \leqq T$$

in a Banach space X. Here T is a constant, $0 \leqq s < T$, $0 \leqq a < 1$, $\mathbf{D}$ is a closed subset of X, z_0 is a constant such that $\rho(A(t)+z_0) \supset \{z \in \mathbf{C}; \operatorname{Re}(z) \leqq 0\}$, $A(t)$ generates an analytic semigroup $e^{-rA(t)}$ for $0 \leqq t \leqq T$, and

$$(A(t)+z_0)^a = (1/\Gamma(1-a)) \int_0^\infty r^{-a}(A(t)+z_0)e^{-r(A(t)+z_0)}\,dr.$$

Our main interest is the Pavel's condition (cf. [7],[8])

$$(1.3)_a\quad \lim_{h\to 0^+} \operatorname{dist}_X(U(t+h,t)v + hf(t,v),\mathbf{D}) = 0, \qquad 0 \leqq t < T, \quad v \in \mathbf{D}_a(t),$$

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where $\mathbf{D}_a(t) = \mathbf{D} \cap D((A(t)+z_0)^a)$, $U(t,s)v$ is the solution of (1.1) with $f = 0$, and $u_0 = v$. We shall show that if

$$U(t,s)\mathbf{D} \subset \mathbf{D} \qquad \text{for } 0 \leqq s < t \leqq T, \tag{1.4}$$

then (1.1)–(1.2) is locally soluble if and only if $(1.3)_a$ is valid. A typical case to which it applies is the semilinear parabolic systems

$$\begin{cases} u_t^k + \sum_{i,j \leqq n} a_{ij}^k(t,x) D_i D_j u^k = f^k(t,x,u,Du), & s < t \leqq T, \quad x \in G, \\ b^k \partial u^k/\partial v + c^k(t,x) u^k = 0, & s < t \leqq T, \quad x \in \partial G, \\ u^k(s,x) = u_0^k(x), & x \in G; \quad k = 1,\ldots,m, \end{cases} \tag{1.5}$$

associated with the flow invariance condition

$$u(t,x) \in I, \qquad s \leqq t < T, \quad x \in G. \tag{1.6}$$

Here G is a smooth bounded domain of R^n, $x = (x_1,\ldots,x_n)$, $D_i = \partial/\partial x_i$, $D = (D_1,\ldots,D_n)$, $u = (u^1,\ldots,u^m)$, I is a closed convex subset of R^m defined in section 2, and v is the outer normal vector field of ∂G. Since the counterpart of (1.4) is satisfied with respect to (1.5)–(1.6), we thus obtain a necessary and sufficient condition for local existence of solutions to the flow invariance problem (1.5)–(1.6).

The flow invariance problem stems from Nagumo [6] in studying ordinary differential equations in R^n. Nagumo's work has now been extended to evolution equations (cf. [1],[3],[5],[7],[8],[9],[10]) and differential inclusions (cf. [2],[12]). The problem (1.1)–(1.2) was studied in [1],[5] under (1.4) and the Nagumo-type condition

$$\liminf_{h \to 0^+} h^{-1} \operatorname{dist}_X(v + hf(t,v), \mathbf{D}) = 0, \qquad 0 \leqq t < T, \quad v \in \mathbf{D}_a(t), \tag{1.7}$$

which is not a necessary condition for solving (1.1)–(1.2). Pavel obtained a result in [7],[8],[9] that $(1.3)_a$ with $a = 0$ is a necessary and sufficient condition for local existence of solutions to (1.1)–(1.2) with $a = 0$. This applies to (1.5)–(1.6) with f independent of Du. We also refer to [1],[3],[10] for existence of solutions to (1.5)–(1.6) under a counterpart of (1.7). Especially, we are interested in [10], where Prüss proved that (1.1)–(1.2) is locally soluble if and only if the following condition holds:

$$(1.8)_a \quad \begin{cases} \text{there is a constant } \sigma > 0 \text{ such that for } 0 \leqq t < T, \\ v \in \mathbf{D}_a(t), \text{ small } h > 0, \text{ there is } w_h \in \mathbf{D}_a(t+h) \\ \text{satisfying } \|z_h\|_X = o(h),\ \|A^a(t+h)z_h\|_X = o(h^\sigma), \text{ where} \\ z_h = U(t+h,t)v + \int_t^{t+h} U(t+h,r)f(t,v)\,dr - w_h. \end{cases}$$

As is announced in [10], whether $(1.3)_a$ is a necessary and sufficient condition for solving (1.1)–(1.2) remains an open question. Our main goal is to give an affirmative answer to such a question with respect to (1.5)–(1.6).

2. Preliminaries and main results

Let X and Y be Banach spaces. Then $L(X;Y)$ denotes the Banach space of all bounded linear operators from X into Y, $X \hookrightarrow Y$ means that X is continuously imbedded in Y, and $X \overset{c}{\hookrightarrow} Y$ means that X is continuously and compactly imbedded in Y. Moreover we set

$$T_\triangle = \{(t,s) \in R^2; 0 \leqq s \leqq t \leqq T\}, \qquad T_\triangle^* = \{(t,s) \in T_\triangle; s < t\},$$

$$J = [0,T], \quad J_{s,r} = J \cap [s, r+\delta], \quad J_{s,r}^* = J \cap (s, s+r], \quad \text{for } s,r > 0.$$

In this paper, we are interested in the following assumptions for (1.1)–(1.2).

(A1) X is a Banach space, and $\{A(t); t \in J\}$ is a family of densely defined and closed linear operators in X for which $\rho(A(t)+z_0) \supset \{z \in \mathbf{C}; \operatorname{Re}(z) \leqq 0\}$, and there is a constant $c_1 > 0$ such that

$$\|(z + A(t))^{-1}\|_X \leqq c_1(1+|z|)^{-1} \qquad \text{for } t \in J, \quad \operatorname{Re}(z) \geqq z_0.$$

(A2) If $D(A(t)) = D(A(0))$ for $t \in J$, then there are constants $c_2 > 0$, $0 < b \leqq 1$ such that

$$\|(A(t) - A(s))(A(s)+z_0)^{-1}\|_X \leqq c_2|t-s|^b \qquad \text{for } t,s \in J.$$

If $D(A(t)) \neq D(A(0))$ for $t \in J$, then $(A(t)+z_0)^{-1}u$, for $u \in X$, is continuously differentiable with respect to t, and there are constants $c_3 > 0$, $1 > -c_4 > 0$ such that

$$\|(A(t)+z_0)(A(t)+z)^{-1}d(A(t)+z_0)^{-1}/dt\|_X \leqq c_3|z-z_0|^{c_4},$$
$$t \in J, \quad \operatorname{Re}(z) \geqq z_0.$$

(A3) $\mathbf{D}$ is a closed subset of X, and $D(A(t)) \overset{c}{\hookrightarrow} X$ for $t \in J$.

(A4) $f: J \times \mathbf{D} \times Y \to X$ is continuous and maps bounded sets into bounded sets, where $Y \hookrightarrow X$ is a Banach space and there is a constant $0 \leqq a_0 < 1$ such that

$$(A(\cdot) + z_0)^{-a} u \in C(J;Y) \qquad \text{for } u \in X, \quad a_0 < a < 1.$$

Definition 2.1. Let $0 \leqq s < T$, and (A1)–(A4) be valid. Then a function u is said to be a local mild solution of (1.1)–(1.2) if there is a constant $\sigma > 0$ such that $u \in C(J_{s,\sigma}; Y)$, $u(J_{s,\sigma}) \subset \mathbf{D}$, and

$$u(t) = U(t,s)u_0 + \int_s^t U(t,r) f(r, u(r))\, dr \qquad \text{for } t \in J_{s,\sigma}.$$

Moreover, we impose the assumptions of (1.5)–(1.6).

(B1) There is a positive constant d such that for $k = 1, \ldots, m$, $c^k \in C^{1+d}(J \times \bar{G}; [0,\infty))$, $b^k \in \{0,1\}$, and $c^k(t,x) = 1$ if $b^k = 0$. If c^k is independent of $t \in J$, then $a^k = (a^k_{ij}): J \times \bar{G} \to R^{n^2}$ is uniformly Hölder continuous, and for $(y_1, \ldots, y_n) \in R^n \setminus \{0\}$, $t \in J$, $x \in \bar{G}$,

$$\sum_{i,j \leqq n} a^k_{ij}(t,x) y_i y_j < 0;$$

otherwise, we additionally suppose that a^k is uniformly continuously differentiable with respect to $t \in J$; $k = 1, \ldots, m$.

(B2) There are $l - 1$ integers $0 = m_0 < m_1 < \cdots < m_l = m$ such that $a^k = a^{m_i}$, $c^k = c^{m_i}$, $b^k = b^{m_i}$ for $m_{i-1} + 1 \leqq k < m_i$, $i = 1, \ldots, l$; $I = I_1 \times \cdots \times I_l$, where I_j is a closed convex subset of $R^{m_{j-1} - m_j}$, and $0 \in I_j$ for $j = 1, \ldots, 1$.

(B3) $f = (f^1, \ldots, f^m): J \times \bar{G} \times I \times R^{nm} \to R^m$ is uniformly Hölder continuous.

Now we state our result for (1.1)–(1.2).

Theorem 2.1. *Let* (A1)–(A4) *be satisfied,* $0 \leqq s < T$, $a_0 < a < 1$, $u_0 \in \mathbf{D}_a(s)$, *and* (1.4) *be valid. Then the problem* (1.1)–(1.2) *admits a local mild solution if and only if* $(1.3)_a$ *holds.*

In order to state our result for (1.5)–(1.6), we set, for $p > n$, and $c_0 > 0$ sufficiently large, $X_p = L^p(G; R^m)$,

$$\mathbf{D}_p = \{u \in X_p; u(x) \in I \text{ for a.e. } x \in G\},$$

$$A_p(t) = \left(\sum_{i,j} a^1_{ij}(t,\cdot) D_i D_j, \ldots, \sum_{i,j} a^m_{ij}(t,\cdot) D_i D_j \right) + c_0$$

with

$$D(A_p(t)) = \{u \in W^{2,p}(G;R^m); b^k \partial u^k/\partial v + c^k(t,x)u^k = 0$$
$$\text{for } x \in \partial G,\ k = 1,\dots,m\},$$

and denote by $U_p: T_\Delta \to L(X_p;X_p)$ the evolution system generated by $\{A_p(t) - c_0; t \in J\}$.

With the above preparations, we state now our result for (1.5)-(1.6).

THEOREM 2.2. *Let* (B1)-(B3) *be satisfied. Then*

$$\operatorname{dist}_{X_p}(U_p(t+h,t)w + hf(t,\cdot,w,Dw),\mathbf{D}_p) = o(h) \qquad (h \searrow 0)$$

for all $p > n$, $0 < q < 1$, $0 \leqq t < T$, $w \in C^{1+q}(\bar{G};R^m)$ *with*

$$b^k \partial w/\partial v + c^k(t,x)w^k = 0, \qquad x \in \partial G, \quad k = 1,\dots,m$$

a necessary and sufficient condition for the validity of the following assertion:

For all $0 \leqq s < T$, $p > n$, $(1 + n/p)/2 < a < 1$, *there exists a constant* $\sigma > 0$ *such that* (1.5)-(1.6) *admits a solution* u *on* $J_{s,\sigma}$ *satisfying* $u(J_{s,\sigma}) \subset I$, $u(t) \in D(A_p(t))$ *for* $t \in J^*_{s,\sigma}$, *and* $u \in C(J_{s,\sigma};Y) \cap C^a(J_{s,\sigma};X_p)$.

3. Proof of the main results

Our theorems are mainly based on [10]. Let us begin with two lemmas.

LEMMA 3.1 (cf. [14], [13], or [4]). *Let* (A1)-(A2) *be valid. Then the operator* $U: T_A \to L(X;X)$ *is the evolution system generated by* $\{A(t); t \in J\}$, *and satisfies, for* $u \in X$, $a \in [0,1]$, $b \in [0,1]$, *and some constant* $c > 0$,

(i) $U(t,s)u \in C(T_\Delta;X)$, $(A(t) + z_0)^b U(t,s)(A(s) + z_0)^{-a} \in C(T^*_\Delta;L(X;X))$,

(ii) $\| (A(t) + z_0)^b U(t,s)(A(s) + z_0)^{-a} \|_X \leqq c(t-s)^{a-b}$, $(t,s) \in T^*_\Delta$,

(iii) $U(t,s)U(s,r) = U(t,r)$, $U(t,t) = \mathrm{id}$, $0 \leqq r \leqq s \leqq t \leqq T$.

LEMMA 3.2 (cf. [10],[1]). *Let* (A1)-(A4), *and* $(1.8)_a$ *with* $a_0 < a < 1$ *be valid. Then for every* $0 \leqq s < T$, *and every* $u_0 \in \mathbf{D}_a(s)$, (1.1)-(1.2) *admits a local mild solution.*

PROOF OF THEOREM 2.1. With the use of Lemma 3.2, we note that to prove the sufficiency, it suffices to prove that $(1.3)_a$-(1.4) implies $(1.8)_a$.

Given $0 \leqq t < T$, and $u \in \mathbf{D}_a(t)$, we have, by $(1.3)_a$, that there is a number $d_0 > 0$ such that

$$\operatorname{dist}_X(U(t+d,t)u + df(t,u),\mathbf{D}) = o(d) \qquad \text{for } d_0 > d > 0,$$

which implies that we can take $v_1 = v_1(d) \in \mathbf{D}$, $v_2 = v_2(d) \in X$ with $\|v_2\|_X = o(d)$ such that

$$U(t+d,t)u + df(t,u) - v_1 - v_2 = 0. \tag{3.1}$$

Let $h > d$ with $h - h^{(1+a)/2a} = d$. We have, by (3.1),

$$U(t+d,t)u + \int_t^{t+h} U(s+d,r)f(t,u)\,dr - v_1 - v_3 = 0, \tag{3.2}$$

where

$$v_3 = v_2 + h^{(1+a)/2a}f(t,u) + \int_t^{t+h} (U(t+d,r)f(t,u) - f(t,u))\,dr,$$

which yields, by (A4) and Lemma 3.1, $\|v_3\|_X = o(h)$. Moreover, applying Lemma 3.1 and using (3.2) with $U(t+h,t+d)$, we have

$$U(t+h,t+d)v_3 = U(t+h,t)u + \int_t^{t+h} U(t+h,r)f(t,u)\,dr - U(t+h,t+d)v_1.$$

Setting $z_h = U(t+h,t+d)v_3$, and $w_h = U(t+h,t+d)v_1$, we have that, by (1.4) and Lemma 3.1, $w_h \in \mathbf{D}_a(t+h)$, and

$$\|z_h\|_X = o(h), \quad \|A^a(t+h)z_h\|_X = h^{-(1+a)/2}O(\|v_3\|_X) = o(h^{(1-a)/2}).$$

Consequently, we have $(1.8)_a$.

To prove the necessity, we note that for $0 \leqq t < T$, $u_0 \in \mathbf{D}_a(t)$, and

$$u(t+h) = U(t+h,t)u_0 + \int_t^{t+h} U(t+h,r)f(r,u(r))\,dr,$$

$$\begin{aligned}
&\operatorname{dist}_X(U(t+h,t)u_0 + hf(t,u_0),\mathbf{D}) \\
&\leqq \|U(t+h,t)u_0 + hf(t,u_0) - u(t+h)\|_X \\
&\leqq \left\|\int_t^{t+h} U(t+h,r)(f(t,u_0) - f(r,u(r))\,dr\right\|_X \\
&\quad + \left\|\int_t^{t+h} (f(t,u_0) - U(t+h,r)f(t,u_0))\,dr\right\|_X \\
&= o(h).
\end{aligned}$$

The proof is complete.

REMARK 3.1. From the proof of Theorem 2.1, the above calculation is still valid if (1.4) is replaced by the assertion that for $0 \leqq t < T$, $u \in \mathbf{D}$, and small $h > 0$, there are $w_h \in D((A(t+h)+z_0)^a)$, and $\sigma > 0$ such that

$$h^{-1}\| U(t+h,t)u - w_h \|_X + h^{-\sigma}\| (A(t+h)+z_0)^a (U(t+h,t)u_h - w_h) \|_X = o(1).$$

PROOF OF THEOREM 2.2. Set $Y = C^1(\bar{G};R^m)$, and for $u \in Y$,

$$f(t,u)(x) = f(t,x,u,Du), \qquad t \in J, \quad x \in \bar{G},$$

$$B(t)u(x) = (b^k \partial u^k/\partial v + c^k(t,x)u^k)_{k=1}^m, \qquad t \in J, \quad x \in \partial G,$$

so that (1.5)-(1.6) can be rewritten in the form in X_p:

$$\text{(3.3)} \qquad \begin{cases} u_t + (A_p(t) - c_0)u = f(t,u), & s < t \leqq T, \\ u(s) = u_0, \quad u(t) \in \mathbf{D}_p, & s \leqq t \leqq T. \end{cases}$$

By making use of a standard calculation (cf. [10],[1]), we have (A1)-(A4) with respect to (3.3), and

$$U(t,s)\mathbf{D}_p \subset \mathbf{D}_p \qquad \text{for } (t,s) \in T_\Delta.$$

From the Sobolev imbedding theorem

$$D(A_p^a(t)) \hookrightarrow \{u \in C^{1+q}(\bar{G};R^m);\ B(t)u = 0 \text{ on } \partial G\}$$

for $1 < 1 + q < 2a - n/p < 2 - n/p$ and $t \in J$, it follows that the condition

$$\text{(3.4)} \qquad \operatorname{dist}_{X_p}(U_p(t+h,t)u + hf(t,u), \mathbf{D}_p) = o(h) \qquad (h \searrow 0)$$

for all $t \in [0,T)$, $1 < 2a - n/p < 2 - n/p$, and $u \in D(A_p^a(t)) \cap \mathbf{D}_p$ is equivalent to the one that (3.4) holds for all $t \in [0,T)$, $1 < 1 + q < 2a - n/p < 2 - n/p$, $u \in C^{1+q}(G;R^m) \cap \mathbf{D}_p$ with $B(t)u = 0$ on ∂G. Hence it remains to prove that each mild solution of (3.3) is, in fact, a solution of (3.3), provided $u_0 \in D(A_p^a(s))$ with $1 < 2a - n/p < 2 - n/p$.

For convenience, we suppose that u is a mild solution of (3.3) with $s = 0$. Hence

$$u(t) = U(t,0)u_0 + \int_0^t U(t,r)f(r,u(r))\,dr, \qquad t \in J.$$

From Lemma 3.1, we obtain immediately that

$$\text{(3.5)} \qquad u \in C^a(J;X_p) \quad \text{and} \quad A_p^a(\cdot)u(\cdot) \in L^\infty(J;X_p).$$

On the other hand, it follows from (B3) that there is a small $\rho > 0$ such that $f(\cdot, u(\cdot)) \in L^\infty(J; C^\rho(\bar{G}; R^m))$. This together with (3.5) and [11] implies that there is a constant $0 < \mu < \min\{\rho, 1/p\}$ such that

$$A_p^\mu(\cdot) f(\cdot, u(\cdot)) \in L^\infty(J; X_p),$$

so that by an elementary calculation, $u(t) \in D(A_p(t))$ for $0 < t \leqq T$. Consequently, u is a solution of (3.3). The proof is complete.

Finally, we give an application of Theorem 2.1 in which **D** is bounded by a function.

EXAMPLE 3.1. Let $1 < 2a - n/p < 2 - n/p$, $b \geqq 1$, $c \geqq 1$,

$$I = \{u \in C^1([0,\pi]; R); u(x) \leqq \sin(x) \text{ for all } x \in [0,\pi]\},$$

$$A_p u = -u_{xx} \qquad \text{with } D(A_p) = W^{2,p}((0,\pi); R) \cap W_0^{1,p}((0,\pi); R),$$

$$f: I \to R \qquad \text{such that } f(u, u_x) = u|u|^{b-1}/(1 + |u_x|^c).$$

Then the problem

$$\text{(3.6)} \qquad \begin{cases} u_t = u_{xx} + f(u, u_x), & x \in (0,\pi), \quad t > 0, \\ u(t,0) = u(t,\pi) = 0, & u(t) \in I, \quad t \geqq 0, \\ u(0,x) = u_0(x), & x \in (0,\pi) \end{cases}$$

admits a unique maximal solution

$$u \in C([0, t_{\max}); D(A_p^a)) \cap C((0, t_{\max}); D(A_p)), u([0, t_{\max})) \subset I.$$

PROOF. Since f is Lipschitz continuous, from the proof of Theorem 2.2 and the local extension procedure it suffices to show that (3.6) admits a local mild solution

$$\text{(3.7)} \qquad u \in C([0,\delta); D(A^a)), \quad u([0,\delta]) \subset I \quad \text{for some } \delta > 0.$$

Let $X = L^p((0,\pi); R)$, $Y = C^1([0,\pi]; R)$, $w(x) = \sin(x)$,

$$\mathbf{D} = \{u \in X; u(x) \leqq w(x) \text{ for a.e. } x \in (0,\pi)\},$$

and e^{-tA} be the analytic semigroup generated by A. Moreover, setting $w_h = e^{-hA}w - w + hAw$, $u_h = e^{-hA}u - w_h$ for small $h > 0$, $u \in \mathbf{D} \cap D(A^a)$, we have

$$\| w_h \|_X + \| A^a w_h \|_X = o(h), \qquad u_h \in \mathbf{D} \cap D(A^a),$$

so that, for $u \in \mathbf{D} \cap D(A^a)$, $h > 0$,

$$e^{-hA}u \leqq e^{-hA}w = w - hw + w_h \leqq w + w_h, \qquad u \in \mathbf{D} \cap D(A^a),$$

$$\begin{aligned} e^{-hA}u + hf(u, u_x) &\leqq e^{-hA}w + hf(u, u_x) \\ &= w + h(-w + f(u, u_x)) + w_h \\ &\leqq w + w_h. \end{aligned}$$

Thus we have

$$\|e^{-hA}u - u_h\|_X + \|A^a(e^{-hA}u - u_h)\|_X = o(h),$$

$$\mathrm{dist}_X(e^{-hA}u + hf(u, u_x), \mathbf{D}) = o(h).$$

It follows from Remark 3.1 and Theorem 2.1 that (3.6) admits a local mild solution u, which obviously satisfies (3.7). The proof is complete.

REMARK 3.2. It should be noted that the function f in (3.6) does not satisfy the Nagumo-type condition (1.7), and it is easy to see that $\{y \in D(A^a); -w \leqq y \leqq w\}$ or $\{y \in D(A^a); 0 \leqq y \leqq w\}$ is also a flow invariant set of (3.6). In a flow invariant set for (1.5) bounded by upper and lower solutions, we refer to [10, Theorem 6]. However, [10, Theorem 6] cannot be applied to (3.6).

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REFERENCES

1. H. Amann, *Invariant sets and existence theorems for semilinear parabolic and elliptic systems*, J. Math. Anal. Appl. **65** (1978), 432–467.

2. J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, 1984.

3. J. W. Bebernes, K. N. Chueh and W. Fulks, *Some applications of invariance for parabolic systems*, Indiana Univ. Math. J. **28** (1979), 269–277.

4. A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

5. R. H. Martin, Jr. and J. H. Lightbourne III, *Relatively continuous nonlinear perturbations of analytic semigroups*, Nonlinear Anal. **1** (1977), 277–292.

6. M. Nagumo, *Über die Lage der Integralkurven gewohnlicher Differentialgleichungen*, Proc. Phys. Math. Soc. Japan **24** (1942), 551–559.

7. N. H. Pavel, *Invariant sets for a class of semilinear equations of evolution*, Nonlinear Anal. **1** (1977), 187–196.

8. N. H. Pavel, *Semilinear equations with dissipative time-dependent domain perturbations*, Isr. J. Math. **46** (1983), 103–122.

9. N. H. Pavel, *Differential Equations, Flow Invariance and Applications*, Res. Notes in Math. 113, Pitman, 1984.

10. J. Prüss, *On semilinear parabolic evolution equations on closed sets*, J. Math. Anal. Appl. **77** (1980), 513–538.

11. R. Seeley, *Interpolation in L^p with boundary conditions*, Studia Math. **44** (1972), 47–60.

12. S. Z. Shi, *Viability theorems for a class of differential-inclusions*, J. Differ. Equ. **79** (1989), 232–257.

13. P. E. Sobolevskii, *Equations of parabolic type in a Banach space*, Am. Math. Soc. Transl., Ser. 2, **49** (1966), 1–62.

14. A. Yagi, *On the abstract linear evolution equations in Banach spaces*, J. Math. Soc. Japan **28** (1976), 290–303.